Matrix Investigation Solution

Part 1

We were given the following matrices:

L = $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ M = $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ and N = $\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$

(a) $L^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$

(b) $M^2 = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 & 12 \\ 12 & 13 \end{pmatrix}$

(c) $N^2 = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 25 & 24 \\ 24 & 25 \end{pmatrix}$

(d) $A^2 = \begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix}\begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix} = \begin{pmatrix} (a+1)^2 + a^2 & a(a+1) + a(a+1) \\ a(a+1) + a(a+1) & (a+1)^2 + a^2 \end{pmatrix}$

$= \begin{pmatrix} 2a^2 + 2a + 1 & 2a^2 + 2a \\ 2a^2 + 2a & 2a^2 + 2a + 1 \end{pmatrix}$

(e) Xabi appears to be right, since we can write A^2 as $\begin{pmatrix} b+1 & b \\ b & b+1 \end{pmatrix}$ where $b = 2a^2 + 2a$

(f) Xavi is always right because the matrix product above in part (d) will work for all values of a

(g) $LM = \begin{pmatrix} 8 & 7 \\ 7 & 8 \end{pmatrix}$

(h) $MN = \begin{pmatrix} 18 & 17 \\ 17 & 18 \end{pmatrix}$

(i) $LN = \begin{pmatrix} 11 & 10 \\ 10 & 11 \end{pmatrix}$

(j) $\begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix}\begin{pmatrix} b+1 & b \\ b & b+1 \end{pmatrix} = \begin{pmatrix} (a+1)(b+1) + ab & b(a+1) + a(b+1) \\ b(a+1) + a(b+1) & (a+1)(b+1) + ab \end{pmatrix}$

$= \begin{pmatrix} a+b+2ab+1 & a+b+2ab \\ a+b+2ab & a+b+2ab+1 \end{pmatrix}$

$\equiv \begin{pmatrix} c+1 & c \\ c & c+1 \end{pmatrix}$ where $c = a + b + 2ab$

(k) $L^3 = \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix}$

(l) $L^4 = \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix}$

(m) Find $L^5 = \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix}$

(n) We notice that the answers to (k), (l) and (m) are also of the form $\begin{pmatrix} b+1 & b \\ b & b+1 \end{pmatrix}$

This would lead to the hypothesis:

For matrices of the form $A = \begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix}$, A^n will be of the form $\begin{pmatrix} k+1 & k \\ k & k+1 \end{pmatrix}$

This actually follows from previous results:

(i) We have shown that, for matrices of the form $A = \begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix}$ matrices of the form A^2 can be written $\begin{pmatrix} b+1 & b \\ b & b+1 \end{pmatrix}$

(ii) A^3 can be written as A^2 x A, and A is of the form $\begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix}$ while A^2 is of the form $\begin{pmatrix} b+1 & b \\ b & b+1 \end{pmatrix}$

(iii) We have shown that such a product - $\begin{pmatrix} a+1 & a \\ a & a+1 \end{pmatrix}\begin{pmatrix} b+1 & b \\ b & b+1 \end{pmatrix}$ - will be of the form $\begin{pmatrix} c+1 & c \\ c & c+1 \end{pmatrix}$

(iv) This means that A^3 will be of the form $\begin{pmatrix} c+1 & c \\ c & c+1 \end{pmatrix}$

(v) A^4 can be written as A^3 x A, and we can iterate through these results again.

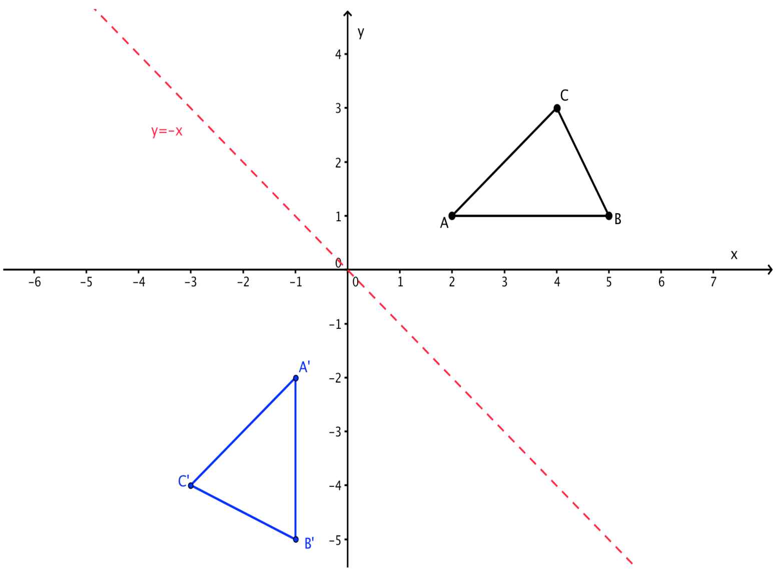
Thus A^n will be of the form $\begin{pmatrix} k+1 & k \\ k & k+1 \end{pmatrix}$

Part 2

(a) MT = $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \end{pmatrix}$ = $\begin{pmatrix} -1 & -1 & -3 \\ -2 & -5 & -4 \end{pmatrix}$

(b) MT gives is the coordinates of the image under the transformation M

(c) M is the transformation: Reflection in the line y = -x



(d) $M^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \implies M^2T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 4 \\ 1 & 1 & 3 \end{pmatrix}$

In other words, M^2 leaves the Triangle unchanged (as expected – it's the identity matrix). Another way of looking at it: the Triangle has been reflected in the line y = -x, then again in the same line, taking us back to the start

(e) $M^3 = M^2M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = M$

$\implies M^3T = \begin{pmatrix} -1 & -1 & -3 \\ -2 & -5 & -4 \end{pmatrix}$

M^3 represents three consecutive reflections in the line y =-x. This is the same as one reflection.

(f) $M^{-1} = \frac{1}{0-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = M$

$\implies M^{-1}T = MT = \begin{pmatrix} -1 & -1 & -3 \\ -2 & -5 & -4 \end{pmatrix}$

This result is to be expected. The inverse of a reflection in the line y= -x is itself the same reflection.

(g) $M^{89998}T$ can be calculated using a GDC (or similar). It can also be calculated "by hand". The best way of calculating it is to use the ***pattern*** already found – namely that:

$M^nT = MT$ for all odd n

and $M^nT = T$ for all even n